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# Finite amplitude instability of plane Couette flow

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The Orr–Sommerfeld equation is solved numerically using expansions in Chebyshev polynomials and the generalized Rayleigh quotient iteration. Accurate results for large values of the parameters are obtained, and these further verify the belief that plane Couette flow is stable to infinitesimal disturbances. For finite disturbances, a formal expansion based on the method of Stuart and Watson as modified by Reynolds & Potter is used. This method shows a transition to instability for a large enough amplitude.

## 1. Introduction

In this paper we are concerned with the question of the stability of plane Couette flow. The case of infinitesimal disturbances has been studied numerically by Gallagher & Mercer (1962), Deardorff (1963) and more recently by Davey (1973). These papers indicate that plane Couette flow is stable for all values of the two parameters: the wavenumber $\alpha$ and the Reynolds number $R$. The first part of this paper develops an efficient method of computing accurate eigenvalues which allows their computation for larger values of the parameters. The eigenvalues obtained further confirm the belief in stability.

However, in a series of experiments, Reichardt (1956) was able to maintain laminar flow only for $R$ up to about 750. The assumption is that nonlinear effects cause the transition to turbulent flow.

The nonlinear analysis centres about an equation for the amplitude of the velocity disturbance of the form $dA/dt = a^{(0)}A + a^{(2)}A^3 + \dots$. Here $a^{(0)} = \alpha c_i$, where $c$ is an eigenvalue of the linearized stability theory. For Couette flow, $a^{(0)} < 0$ for all values of $\alpha$ and $R$. If $a^{(2)}$ is positive, then for a large enough amplitude $dA/dt$ will be positive, and disturbances will grow instead of dying out.

The second part of this paper involves numerical calculations of the values of $A$ for which $a^{(0)}A + a^{(2)}A^3 = 0$. This gives the threshold amplitude for a second-order approximation, i.e. the amplitude for which the disturbance neither grows nor decays. These threshold amplitudes have been calculated for some special cases by Ellingsen, Gjevik & Palm (1970). Also, Davey & Nguyen (1971) discussed this problem for the similar case of pipe flow. This paper presents more detailed calculations.

Also, the accuracy of this second-order approximation has been questioned. So a fourth-order approximation has been calculated for a few cases. These results are fairly close to the second-order approximations, and are further destabilizing.

## 2. The linear problem

We consider the flow of a viscous incompressible fluid between two horizontal planes. In non-dimensional form, the planes are two units apart and move such that the speed of the upper plane is $+1$ and the speed of the lower plane is $-1$. The $x$ axis is chosen midway between the planes. The Reynolds number $R$ is the reciprocal of the kinematic viscosity, and $u(x,y,t)$ and $v(x,y,t)$ are the $x$ and $y$ components of the fluid velocity. The basic laminar flow is $u = y$, $v = 0$. We shall study perturbations of the form $u = y+u'$, $v = v'$, where $u'$ and $v'$ are small.

We may introduce a perturbation stream function $\psi$ defined by $u' = \partial\psi/\partial y$ and $v' = -\partial\psi/\partial x$ and seek periodic solutions of the form $\psi = \phi(y)\exp\{i\alpha(x-ct)\}$. Substituting into the Navier–Stokes equations and neglecting second-order quantities, we obtain the familiar Orr–Sommerfeld equation

$$\{(D^2-\alpha^2)^2 - i\alpha R(y-c)(D^2-\alpha^2)\}\phi = 0 \tag{2.1}$$

with boundary conditions

$$\phi(\pm 1) = D\phi(\pm 1) = 0. \tag{2.2}$$

In this formulation, $\psi$ is a complex-valued function. But since the original Navier–Stokes equations contain no complex quantities, and have been linearized, $\frac{1}{2}(\psi+\bar{\psi})$ is also a solution, and is real valued.

For given values of $\alpha$ and $R$, a non-trivial solution for $\phi$ exists only when $c = c_r + ic_i$ is a complex eigenvalue. If $c_i > 0$ the flow is unstable and if $c_i < 0$ the flow is stable.

We shall approximate $\phi(y)$ by

$$\sum_{n=0}^{N} a_n T_n(y),$$

where $T_n(y)$ is the $n$th-degree Chebyshev polynomial, defined by $T_n(\cos\theta) = \cos n\theta$. The advantages of Chebyshev polynomials are discussed in general by Fox & Parker (1972) and for the particular case of plane parallel flows by Orszag (1971). The most important fact is that Chebyshev polynomial approximations are of infinite order, in the sense that errors decrease more rapidly than any power of $1/N$ as $N$ approaches infinity. So accurate solutions can be obtained more rapidly using Chebyshev polynomials.

Also, the equations for the coefficients $a_n$ can be obtained relatively easily using the relations

$$2\int T_n = \frac{c_n T_{n+1}}{n+1} - \frac{d_{n-2} T_{n-1}}{n-1} \tag{2.3}$$

and

$$yT_n = \tfrac{1}{2}(d_{n-1}T_{n-1} + c_n T_{n+1}), \tag{2.4}$$

where $c_n = 0$ if $n < 0$, $c_0 = 2$, $c_n = 1$ if $n > 0$, and $d_n = 0$ if $n < 0$, $d_n = 1$ if $n \geqslant 0$. It is more efficient to integrate (2.1) four times to obtain

$$\phi - 2\alpha^2\iint\phi + \alpha^4\iiiint\phi - i\alpha R\{\iint y\phi - 2\iiint\phi - \alpha^2\iiiint y\phi - c\iint\phi + \alpha^2 c\iiiint\phi\} = 0. \tag{2.5}$$

If we represent the $k$th integral of $\phi$ by $\Sigma\, a_n^k T_n$ and the $k$th integral of $y\phi$ by $\Sigma\, b_n^k T_n$, this leads to a system of equations

$$ia_n - 2i\alpha^2 a_n^2 + i\alpha^4 a_n^4 + \alpha R b_n^2 - 2\alpha R a_n^3 - \alpha^3 R b_n^4 - c\alpha R a_n^2 + c\alpha^3 R a_n^4 = 0. \tag{2.6}$$

If we let $\Sigma^2$ indicate that the summation is in steps of two, then

$$a_n^2 = \sum_{k=1}^{3}{}^2 \sum_{h=1}^{3}{}^2 \frac{i^{k+h-2} a_{n-4+k+h}\, c_{n-4+k+h}}{4n(n-2+k)},$$

$$a_n^3 = \sum_{l=1}^{3}{}^2 \sum_{k=1}^{3}{}^2 \sum_{h=1}^{3}{}^2 \frac{i^{k+h+l+1} a_{n-6+h+k+l}\, c_{n-6+h+k+l}}{8n(n-2+l)(n-4+k+l)},$$

$$a_n^4 = \sum_{m=1}^{3}{}^2 \sum_{l=1}^{3}{}^2 \sum_{k=1}^{3}{}^2 \sum_{h=1}^{3}{}^2 \frac{i^{k+h+l+m} a_{n-8+k+h+l+m}\, c_{n-8+k+h+l+m}}{16n(n-2+m)(n-4+m+l)(n-6+m+l+k)},$$

$$b_n^2 = \sum_{j=1}^{3}{}^2 \sum_{k=1}^{3}{}^2 \sum_{h=1}^{3}{}^2 \frac{i^{k+h-2} a_{n-6+k+h+j}\, c_{n-6+k+h+j}}{8n(n-2+k)},$$

$$b_n^4 = \sum_{j=1}^{3}{}^2 \sum_{m=1}^{3}{}^2 \sum_{l=1}^{3}{}^2 \sum_{k=1}^{3}{}^2 \sum_{h=1}^{3}{}^2 \frac{i^{m+l+k+h} a_{n-10+m+l+k+h+j}\, c_{n-10+m+l+k+h+j}}{32n(n-2+m)(n-4+m+l)(n-6+m+l+k)}$$

(if $n = 4, j = 3$ and $m = l = k = h = 1$, the above coefficient is doubled).

Because of the integration, (2.6) contains arbitrary constants for the cases $n = 0$, 1, 2 and 3. So we replace the equations by equations representing the boundary conditions. Using the relations $T_n(\pm 1) = (\pm 1)^n$ and $DT_n(\pm 1) = n^2(\pm 1)^{n-1}$, we can obtain the equations

$$\sum_{n=0}^{N}{}^2 a_n = 0, \quad \sum_{n=1}^{N}{}^2 a_n = 0, \quad \sum_{n=0}^{N}{}^2 n^2 a_n = 0, \quad \sum_{n=1}^{N}{}^2 n^2 a_n = 0.$$

Using these four equations plus (2.6) for $n = 4, 5, \ldots, N$, we obtain a system of $N+1$ equations in $N+1$ unknowns. In matrix form this can be written as

$$\mathbf{XU} - c\mathbf{YU} = 0,$$

where $\mathbf{X}$ is a complex-valued matrix, $\mathbf{Y}$ is a real-valued matrix and $\mathbf{U} = (a_0, a_1, \ldots, a_N)^{\mathrm{T}}$ is the eigenvector to be determined.

## 3. Solving the matrix eigenvalue equation

A standard method of solving the above system would be the LR or QR matrix eigenvalue algorithm, described by Wilkinson (1965) and modified for problems of this type by Gary & Helgason (1970). However, this method finds all the eigenvalues, and we are mainly interested in the eigenvalue whose imaginary part is closest to zero. So an alternative method is used, based on a series of papers by Ostrowski (1958–9) discussing the Rayleigh quotient.

We define a generalized Rayleigh quotient $R(\mathbf{U}, \mathbf{V}) = (\mathbf{V}^{\mathrm{T}}\mathbf{XU})/(\mathbf{V}^{\mathrm{T}}\mathbf{YU})$. If

$$\mathbf{XU} = c\mathbf{YU} \quad \text{and} \quad \mathbf{V}^{\mathrm{T}}\mathbf{X} = c\mathbf{V}^{\mathrm{T}}\mathbf{Y},$$

then $R(\mathbf{U}, \mathbf{V}) = c$. Moreover, the quotient is stable, i.e. $R(\mathbf{U}+\epsilon\mathbf{U}', \mathbf{V}+\epsilon\mathbf{V}')$ differs from $R(\mathbf{U}, \mathbf{V})$ only by terms of second order in $\epsilon$.

This makes possible an iterative procedure for determining any specific eigenvalue $c$. We start with an approximation $c_0$ and arbitrary vectors $\mathbf{U}_0$ and $\mathbf{V}_0$. Here $c_0$ can be obtained from the asymptotic analysis, by using an eigenvalue computed for nearby

| $X$ | $Y$ | $Z$ |
|---|---|---|
| 0·1 | 2·874 | −8·065 |
| 0·2 | 2·530 | −6·012 |
| 0·3 | 2·357 | −4·978 |
| 0·4 | 2·254 | −4·338 |
| 0·5 | 2·199 | −3·900 |
| 0·6 | 2·178 | −3·576 |
| 0·7 | 2·182 | −3·324 |
| 0·8 | 2·203 | −3·122 |
| 0·9 | 2·236 | −2·943 |
| 1·0 | 2·280 | −2·813 |

TABLE 1. Values of $X$, $Y$ and $Z$ for $R \geqslant 200$.

values of $\alpha$ and $R$ or by using a matrix algorithm for a smaller value of $N$. The iteration is defined by

$$(\mathbf{X}-c_k\mathbf{Y})\,\mathbf{U}_{k+1} = \mathbf{Y}\mathbf{U}_k, \quad \mathbf{V}_{k+1}^{\mathrm{T}}(\mathbf{X}-c_k\mathbf{Y}) = \mathbf{V}_k^{\mathrm{T}}\mathbf{Y}, \tag{3.1}$$

$$c_{k+1} = R(\mathbf{U}_{k+1}, \mathbf{V}_{k+1}). \tag{3.2}$$

The first two equations define an inverse iteration for obtaining eigenfunctions. This is a standard procedure if the eigenvalue is known. The convergence of the entire iteration is cubic, i.e. $(c_{k+1}-c)/(c_k-c)^3$ approaches a constant.

The matrix $\mathbf{X}-c_k\mathbf{Y}$ is decomposed into the product of a lower triangular matrix and an upper triangular matrix. The same decomposition can be used in obtaining both $\mathbf{U}$ and $\mathbf{V}$. For the first several iterations only (3.1) are used. This allows the eigenvectors to converge relatively closely to the correct values. Generally, the eigenvalue will be accurate to four decimal places after three or four full iterations.

The generalized Rayleigh quotient iteration is much more efficient than a matrix method, particularly for large $N$. For example, if $N = 70$, the above iteration is more than five times as fast as the LR algorithm.

## 4. Numerical results for the linear case

We are interested in whether or not plane Couette flow is stable with respect to infinitesimal disturbances. However, it is not easy to cover systematically the entire $\alpha$, $R$ plane. Also, for $\alpha$ and $R$ large, where instability is more likely to occur, the Orr–Sommerfeld equation is difficult to solve.

The most recent results on plane Couette flow are due to Davey (1973). Using asymptotic analysis, he showed that, for $(\alpha R)^{\frac{1}{3}}$ large, if $R^{\frac{1}{2}}$ is much larger than $\alpha$ then

$$c_r \approx 1-4{\cdot}1287/(\alpha R)^{\frac{1}{3}}, \tag{4.1a}$$

$$c_i \approx -\alpha/R-1{\cdot}0625/(\alpha R)^{\frac{1}{3}} \tag{4.1b}$$

and if $\alpha$ is much larger than $R^{\frac{1}{2}}$ then

$$c_r \approx 1-2{\cdot}0249/(\alpha R)^{\frac{1}{3}}, \tag{4.2a}$$

$$c_i \approx -\alpha/R-1{\cdot}1691/(\alpha R)^{\frac{1}{3}}. \tag{4.2b}$$

If we introduce new variables $X = \alpha R^{-\frac{1}{3}}$ and $Y = -R^{\frac{1}{3}}c_i$, then (4.1$b$) becomes

$$Y \approx X + 1{\cdot}0625X^{-\frac{1}{2}} \tag{4.3}$$

and (4.2$b$) becomes

$$Y \approx X + 1{\cdot}1691X^{-\frac{1}{2}}. \tag{4.4}$$

This suggests the possibility that $Y$ depends only on $X$, even when $\alpha$ and $R^{\frac{1}{3}}$ are of the same order of magnitude and the asymptotic analysis does not hold.

Davey claims that this is the case for $R \geqslant 200$, and that this has been checked for values of $\alpha R$ up to 100 000. To verify this result, eigenvalues were computed for values of $R$ up to 5000 and values of $\alpha R$ up to 250 000. In these cases $Y$ did depend only on $X$ up to four decimal places. Table 1 shows the relationship between $X$ and $Y$. If we let $Z = R^{\frac{1}{3}}(c_r - 1)$, we have a similar relationship for the real part of the eigenvalue.

For $R \geqslant 200$, the minimum value of $Y$ occurs at $X = 0{\cdot}63$, $Y = 2{\cdot}17725$. Here $c_i = -2{\cdot}17725R^{-\frac{1}{3}}$, indicating that the flow is stable.

## 5. The nonlinear analysis

Since the linear results do not correspond with the experimental evidence, we shall attempt to find a nonlinear solution. The main approach to this type of problem was developed by J. T. Stuart and J. Watson in a series of papers in the early 1960s. We shall use the notation of Reynolds & Potter (1967), who modified this approach.

We begin with the Navier–Stokes equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0, \tag{5.1$a$}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0, \tag{5.1$b$}$$

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{5.1$c$}$$

with boundary conditions

$$u(x, \pm 1, t) = \pm 1, \quad v(x, \pm 1, t) = 0. \tag{5.2}$$

We attempt to solve this system in terms of the linear stream function

$$\psi(x, y, t) = \phi(y) \exp\{i\alpha(x - ct)\}$$

and its harmonics. It is useful to introduce a change of variables $\theta = \alpha x + \omega t$, $\omega = \omega(A)$, $A = A(t)$. Here $\theta$ represents the periodic part of the solution. $\omega$ is the frequency, which in general will depend on the size of the disturbance. $A(t)$ is an amplitude function yet to be defined. In the linear case $A(t) = \exp(\alpha c_i t)$.

We introduce a stream function defined by $\partial\psi/\partial y = \alpha u$ and $\partial\psi/\partial\theta = -v$. Eliminating the pressure $p$ and using the notation $\zeta = \partial^2\psi/\partial y^2 + \alpha^2\partial^2\psi/\partial\theta^2$, the Navier–Stokes equations become

$$\frac{dA}{dt}\frac{\partial \zeta}{\partial A} + \left[\omega + t\frac{d\omega}{dA}\frac{dA}{dt} + \frac{\partial\psi}{\partial y}\right]\frac{\partial\zeta}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial y} - \frac{1}{R}\left[\frac{\partial^2\zeta}{\partial y^2} + \alpha^2\frac{\partial^2\zeta}{\partial\theta^2}\right] = 0. \tag{5.3}$$

We can now approximate the stream function by a truncated Fourier series

$$\psi(A, y, \theta) = \sum_{k=0}^{K} [\psi^{(k)}(A, y)\, e^{ik\theta} + \tilde{\psi}^{(k)}(A, y)\, e^{-ik\theta}]. \tag{5.4}$$

Using the notation $Z^{(k)} = \partial^2\psi^{(k)}/\partial y^2 - \alpha^2k^2\psi^{(k)}$, (5.3) becomes the set of equations

$$\frac{dA}{dt}\frac{\partial Z^{(k)}}{\partial A} + \left[\omega + t\frac{dA}{dt}\frac{d\omega}{dA}\right] ikZ^{(k)} - \frac{1}{R}\left\{\frac{\partial^2}{\partial y^2} - \alpha^2k^2\right\} Z^{(k)} + \frac{i}{1+\delta_{k0}}$$
$$\times\left\{\sum_{j=0}^{k} j\left[Z^{(j)}\frac{\partial\psi^{(k-j)}}{\partial y} - \psi^{(j)}\frac{\partial Z^{(k-j)}}{\partial y}\right] + \sum_{j=k}^{K} j\left[Z^{(j)}\frac{\partial\tilde{\psi}^{(j-k)}}{\partial y} - \psi^{(j)}\frac{\partial\tilde{Z}^{(j-k)}}{\partial y}\right]\right.$$
$$\left. + \sum_{j=0}^{K-k} j\left[-\tilde{Z}^{(j)}\frac{\partial\psi^{(k+j)}}{\partial y} + \tilde{\psi}^{(j)}\frac{\partial Z^{(k+j)}}{\partial y}\right]\right\} = 0. \tag{5.5}$$

If the amplitude is small, we can seek a solution as a power series in $A$. So we make the approximations

$$\psi^{(k)} = \sum_{n=k}^{K} A^n\phi^{(k,n)}(y), \tag{5.6}$$

$$\frac{1}{A}\frac{dA}{dt} = \sum_{n=0}^{K} A^n a^{(n)}, \tag{5.7}$$

$$\omega + t\frac{d\omega}{dA}\frac{dA}{dt} = \sum_{n=0}^{K} A^n b^{(n)}. \tag{5.8}$$

If $A = 0$ there is no disturbance, and we have the laminar case. Since $\phi^{(0;0)}$ is the only term of the expansion (5.6) in this case, $D\phi^{(0;0)} = \frac{1}{2}\alpha y$.

Substituting these expansions into (5.5), we get a series of equations

$$L_{kn}\,\phi^{(k;n)} = i\alpha c^{(n-1)}G\delta_{k1} + H_{kn} \tag{5.9}$$

with boundary conditions

$$\phi^{(k;n)}(\pm 1) = D\phi^{(k;n)}(\pm 1) = 0, \tag{5.10}$$

where

$$L_{kn} = ik[(-ink^{-1}a^{(0)} + b^{(0)} + \alpha y)(D^2 - \alpha^2k^2)] - R^{-1}(D^2 - \alpha^2k^2)^2, \tag{5.11}$$

$$i\alpha c^{(n)} = -a^{(n)} - ib^{(n)}, \tag{5.12}$$

$$G = (D^2 - \alpha^2)\,\phi^{(1;1)}, \tag{5.13}$$

$$z^{(k;n)} = (D^2 - \alpha^2k^2)\,\phi^{(k;n)}, \tag{5.14}$$

$$H_{kn} = -\sum_{m=k}^{n-1}(ma^{(n-m)} + ikb^{(n-m)})\,z^{(k;m)} + \frac{F_{kn}}{1+\delta_{k0}} \tag{5.15}$$

(if $k = 1$, the above sum starts at $m = 2$) and

$$F_{kn} = \sum_{j=0}^{k}\sum_{m=j}^{n-k+j} ij[z^{(j;m)}D\phi^{(k-j;\,n-m)} - \phi^{(j;\,m)}Dz^{(k-j;\,n-m)}]$$
$$+ \sum_{j=k}^{K}\sum_{m=j}^{n-j+k} ij[z^{(j;\,m)}D\tilde{\phi}^{(j-k;\,n-m)} - \phi^{(j;\,m)}D\tilde{z}^{(j-k;\,n-m)}]$$
$$+ \sum_{j=0}^{K-k}\sum_{m=j}^{n-k-j} ij[-\tilde{z}^{(j;\,m)}D\phi^{(k+j;\,n-m)} + \tilde{\phi}^{(j;\,m)}Dz^{(k+j;\,n-m)}] \tag{5.16}$$

(if $j = k$ and $m = n$ in $F_{kn}$, that term is deleted).

If $k = n = 1$, (5.9) becomes

$$\{(a^{(0)} + ib^{(0)} + i\alpha y)(D^2 - \alpha^2) - R^{-1}(D^2 - \alpha^2)^2\}\,\phi^{(1;1)} = 0. \tag{5.17}$$

This is the Orr–Sommerfeld equation with $b^{(0)} = -\alpha c_r$ and $a^{(0)} = \alpha c_i$. The amplitude of $\phi^{(1;1)}$ is not determined by the Orr–Sommerfeld equation. We shall use the standard normalization $\phi^{(1;1)}(0) = 1$. This will implicitly determine the function $A(t)$, which is the other part of the solution relating to the size of the disturbance.

Returning to the set of equations (5.11), $\phi^{(0;1)}$ can be determined explicitly and is equal to zero. The right sides of the equations for $\phi^{(0;2)}$ and $\phi^{(2;2)}$ are functions of $\phi^{(1;1)}$, so we have inhomogeneous linear equations for these functions. But to continue the calculations we need to determine the constants $c^{(n)}$. This will require some additional assumption.

This would not be a problem if $a^{(0)} = \alpha c_i = 0$. In this case $L_{1n}$ would be just the Orr–Sommerfeld operator. So the adjoint homogeneous equation would have a solution $\xi(y)$, and (5.11) would not have a solution unless its right side was orthogonal to $\xi$. This consistency condition would determine the $c^{(n)}$.

This can be done in the similar case of plane Poiseuille flow. Here there is a linear stability curve in the $\alpha, R$ plane on which $c_i = 0$. On this curve, the constants are determined by the orthogonality condition. Near the neutral curve the same method is used on the basis of continuity. The $c^{(n)}$ vanish for $n$ odd, and

$$dA/dt = a^{(2)}A^3 + a^{(4)}A^5 + \dots.$$

So the sign of $a^{(2)}$ determines the stability of the flow.

For Couette flow, $a^{(0)}$ is never zero. So instead of insisting that the linear flow be at the transition point, we shall assume that the nonlinear flow is at the transition point, i.e. $dA/dt = 0$. So $A(t)$ will be a constant, and any solution will be a steady-state solution with frequency $\omega$. Then smaller amplitudes will correspond to stability, and larger amplitudes to instability.

In this case, we expand

$$\omega = \sum_{n=0}^{K} \omega^{(n)} A^n.$$

Although $\omega$ is a real number we shall view it as complex as a mathematical convenience. The physical problem will have solutions for only certain values of $A$. We obtain a set of equations similar to (5.9):

$$\mathscr{L}_{kn}\,\phi^{(k;n)} = -i\omega^{(n-1)} G\delta_{k1} + \mathscr{H}_{kn}, \tag{5.18}$$

where

$$\mathscr{L}_{kn} = ik[(\omega^{(0)} + \alpha y)(D^2 - \alpha^2 k^2)] - R^{-1}(D^2 - \alpha^2 k^2)^2, \tag{5.19}$$

$$\mathscr{H}_{kn} = -ik \sum_{m=k}^{n-1} \omega^{(n-m)} z^{(k;m)} + \frac{F_{kn}}{1+\delta_{k0}} \tag{5.20}$$

(if $k = 1$, the above sum starts at $m = 2$).

The problem for $k = n = 1$ is again the Orr–Sommerfeld equation, with a complex $\omega^{(0)}$ replacing $-\alpha c$. Since the operators $\mathscr{L}_{1n}$ are all identical with $\mathscr{L}_{11}$, the homogeneous equations will all possess the eigensolution $\phi^{(1;1)}$, and the $\omega^{(n)}$ must be selected such that the adjoint orthogonality condition is satisfied. So

$$i\omega^{(n-1)} = \int_{-1}^{1} \mathscr{H}_{1n}\,\xi\,dy \Big/ \int_{-1}^{1} G\xi\,dy.$$

The solutions of the physical problem are found by selecting the values of $A$ for which $\omega$ is real. These are given by the roots of the equation

$$\sum_{n=0}^{K} A^n \omega_i^{(n)} = 0.$$

The consistency condition results in $\omega^{(n)} = 0$ for $n$ odd and $\phi^{(k;\,m)} = 0$ for $k+n$ odd.

We shall attempt a second-order solution, i.e. $\omega = \omega^{(0)} + \omega^{(2)}A^2$. However, since the steady-state solution we obtain is not an asymptotic solution, it is questionable how accurate the approximation is. This is discussed in Davey & Nguyen (1971) and Ellingsen *et al.* (1970). While the solution is not expected to be particularly accurate, it should at least give an indication of the nonlinear behaviour.

In order to obtain a better idea of the accuracy, we have calculated $\omega^{(4)}$ for some parameter values. This is much more difficult to solve for numerically, so we only have solutions for $\alpha = 1$ and $R$ small. However, the results here are reasonably close to the second-order approximations.

## 6. Method of solution

As in the linear case, we approximate functions by a sum of Chebyshev polynomials. $\omega^{(0)} = -\alpha c$ and $\phi^{(1;\,1)}$ are computed as before.

The adjoint homogeneous system has the form

$$\left.\begin{aligned} \{(D^2-\alpha^2)^2 - i\alpha R[(y+\omega^{(0)}/\alpha)(D^2-\alpha^2)+2D]\}\,\xi = 0,\\ \xi(\pm 1) = D\xi(\pm 1) = 0. \end{aligned}\right\} \tag{6.1}$$

This is integrated four times and multiplied by $i$. Letting

$$\xi(y) = \sum_{n=0}^{N} a_n T_n(y)$$

and substituting, we obtain the system of equations

$$ia_n - 2i\alpha^2 a_n^2 + i\alpha^4 a_n^4 + \alpha R b_n^2 - \alpha^3 R b_n^4 - \alpha R c a_n^2 + \alpha^3 R c a_n^4 = 0. \tag{6.2}$$

$\xi$ is then computed using the same inverse iteration, except that the eigenvalue is now kept fixed.

To solve the inhomogeneous equations for $\phi^{(0;\,2)}$ and $\phi^{(2;\,2)}$, we need subroutines for taking derivatives and integrals and for multiplying. If we let

$$D\left[\sum_{n=0}^{N} a_n T_n\right] = \sum_{n=0}^{N-1} a_n^d T_n,$$

then by (2.3)

$$c_n a_n^d = 2 \sum_{k=n+1}^{N} k a_k.$$

Also $a_n^1 = (a_{n-1}c_{n-1} - a_{n+1})/2n$. Multiplication can be done by using the relation

$$T_r T_s = \tfrac{1}{2}[T_{r+s} + T_{|r-s|}]. \tag{6.3}$$

If

$$\phi = \sum_{n=0}^{N} a_n T_n, \quad \psi = \sum_{n=0}^{N} b_n T_n,$$

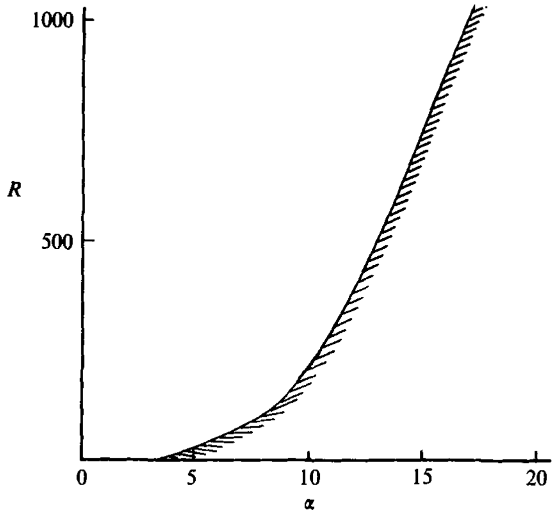


FIGURE 1. Location of steady-state solutions.

then

$$\phi\psi = \sum_{i=0}^{N}\sum_{j=0}^{N}\frac{a_i b_j}{2}[T_{i+j}+T_{|i-j|}].$$

Truncating gives

$$d_n = \frac{1}{2}\left\{\sum_{k=0}^{N} a_k b_{n-k} + \frac{1}{c_n}\sum_{k=n}^{N} a_k b_{k-n} + \frac{1}{c_n}\sum_{k=0}^{N-n} a_k b_{k+n}\right\}.$$

Solving for $\omega^{(2)}$ also requires evaluating definite integrals. Using the formula $T_n(\pm 1) = (\pm 1)^n$,

$$\int_{-1}^{1}\sum_{n=0}^{N} a_n T_n = 2a_0 + \sum_{n=2}^{N}{}^{2}\frac{-2a_n}{n^2-1}.$$

For a second-order approximation, $\omega = \omega^{(0)} + A^2\omega^{(2)}$. Since $\omega$ is real, $0 = \omega_i^{(0)} + A^2\omega_i^{(2)}$, which determines $A$.

For a fourth-order approximation, we also need to solve for $\phi^{(1;\,3)}$, $\phi^{(3;\,3)}$, $\phi^{(0;\,4)}$ and $\phi^{(2;\,4)}$. Here $\phi^{(1;\,3)}$ is not determined uniquely, since we can add any multiple of $\phi^{(1;\,1)}$. We use the normalization $\phi^{(1;\,3)}(0) = 1$. $A$ is determined by solving the equation $0 = \omega_i^{(0)} + A^2\omega_i^{(2)} + A^4\omega_i^{(4)}$.

## 7. Numerical results

The above procedure was carried out for a variety of values for $\alpha$ and $R$. The results are summarized in figure 1. Steady-state solutions do exist for values of $\alpha$ and $R$ to the left of the given curve, but there are no such solutions for values to the right of the curve.

Table 2 shows the amplitude $A$ and the frequency $\omega$ associated with the solutions. The column for $-\alpha c_r$ represents the linear frequency. $E$ measures the size of the disturbance, where we define $E$ by

$$E \equiv \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}(|u'|^2 + |v'|^2)\,d\theta\,dy,$$

| $R$ | $\alpha$ | $-\alpha c_r$ | $A$ | $\omega$ | E |
|---|---|---|---|---|---|
| 100 | 1 | $-2{\cdot}07 \times 10^{-1}$ | $5{\cdot}21 \times 10^{-2}$ | $-2{\cdot}51 \times 10^{-1}$ | $1{\cdot}26 \times 10^{-2}$ |
| 100 | 2 | $-7{\cdot}98 \times 10^{-1}$ | $3{\cdot}99 \times 10^{-2}$ | $-4{\cdot}61 \times 10^{-1}$ | $1{\cdot}09 \times 10^{-2}$ |
| 100 | 4 | $-2{\cdot}26$ | $1{\cdot}73 \times 10^{-2}$ | $-3{\cdot}49 \times 10^{-1}$ | $2{\cdot}53 \times 10^{-2}$ |
| 100 | 6 | $-3{\cdot}85$ | $1{\cdot}01 \times 10^{-2}$ | $4{\cdot}45$ | $1{\cdot}11 \times 10^{-1}$ |
| 100 | 7 | $-4{\cdot}67$ | $1{\cdot}72 \times 10^{-2}$ | $8{\cdot}64 \times 10^{1}$ | $1{\cdot}33$ |
| 200 | 1 | $-3{\cdot}50 \times 10^{-1}$ | $2{\cdot}55 \times 10^{-2}$ | $-3{\cdot}78 \times 10^{-1}$ | $4{\cdot}98 \times 10^{-3}$ |
| 200 | 2 | $-1{\cdot}01$ | $1{\cdot}71 \times 10^{-2}$ | $-8{\cdot}48 \times 10^{-1}$ | $3{\cdot}73 \times 10^{-3}$ |
| 200 | 3 | $-1{\cdot}76$ | $1{\cdot}05 \times 10^{-2}$ | $-1{\cdot}32$ | $3{\cdot}96 \times 10^{-3}$ |
| 200 | 4 | $-2{\cdot}55$ | $6{\cdot}20 \times 10^{-3}$ | $-1{\cdot}66$ | $5{\cdot}03 \times 10^{-3}$ |
| 200 | 5 | $-3{\cdot}37$ | $3{\cdot}65 \times 10^{-3}$ | $-1{\cdot}71$ | $7{\cdot}28 \times 10^{-3}$ |
| 200 | 6 | $-4{\cdot}21$ | $2{\cdot}18 \times 10^{-3}$ | $-1{\cdot}24$ | $1{\cdot}15 \times 10^{-2}$ |
| 200 | 7 | $-5{\cdot}06$ | $1{\cdot}35 \times 10^{-3}$ | $4{\cdot}84 \times 10^{-1}$ | $2{\cdot}04 \times 10^{-2}$ |
| 200 | 8 | $-5{\cdot}92$ | $9{\cdot}35 \times 10^{-4}$ | $7{\cdot}19$ | $4{\cdot}74 \times 10^{-2}$ |
| 200 | 9 | $-6{\cdot}79$ | $4{\cdot}61 \times 10^{-3}$ | $1{\cdot}57 \times 10^{3}$ | $5{\cdot}75$ |
| 500 | 1 | $-5{\cdot}09 \times 10^{-1}$ | $1{\cdot}03 \times 10^{-2}$ | $-5{\cdot}33 \times 10^{-1}$ | $1{\cdot}43 \times 10^{-3}$ |
| 500 | 4 | $-2{\cdot}87$ | $1{\cdot}99 \times 10^{-3}$ | $-2{\cdot}53$ | $9{\cdot}68 \times 10^{-4}$ |
| 500 | 6 | $-4{\cdot}59$ | $5{\cdot}01 \times 10^{-4}$ | $-3{\cdot}58$ | $1{\cdot}30 \times 10^{-3}$ |
| 500 | 9 | $-7{\cdot}26$ | $6{\cdot}49 \times 10^{-5}$ | $-3{\cdot}59$ | $2{\cdot}99 \times 10^{-3}$ |
| 500 | 12 | $-9{\cdot}98$ | $1{\cdot}42 \times 10^{-5}$ | $2{\cdot}32 \times 10^{1}$ | $2{\cdot}41 \times 10^{-2}$ |
| 700 | 1 | $-5{\cdot}58 \times 10^{-1}$ | $7{\cdot}70 \times 10^{-3}$ | $-5{\cdot}81 \times 10^{-1}$ | $9{\cdot}24 \times 10^{-4}$ |
| 700 | 3 | $-2{\cdot}13$ | $2{\cdot}82 \times 10^{-3}$ | $-2{\cdot}00$ | $5{\cdot}95 \times 10^{-4}$ |
| 700 | 6 | $-4{\cdot}71$ | $3{\cdot}16 \times 10^{-4}$ | $-4{\cdot}04$ | $6{\cdot}57 \times 10^{-4}$ |
| 700 | 8 | $-6{\cdot}50$ | $7{\cdot}26 \times 10^{-5}$ | $-5{\cdot}03$ | $9{\cdot}34 \times 10^{-4}$ |
| 700 | 9 | $-7{\cdot}41$ | $3{\cdot}51 \times 10^{-5}$ | $-5{\cdot}26$ | $1{\cdot}18 \times 10^{-3}$ |
| 700 | 12 | $-1{\cdot}01 \times 10^{1}$ | $4{\cdot}62 \times 10^{-6}$ | $-2{\cdot}00$ | $3{\cdot}62 \times 10^{-3}$ |
| 700 | 14 | $-1{\cdot}20 \times 10^{1}$ | $2{\cdot}21 \times 10^{-6}$ | $4{\cdot}76 \times 10^{1}$ | $2{\cdot}46 \times 10^{-2}$ |
| 1000 | 1 | $-6{\cdot}05 \times 10^{-1}$ | $5{\cdot}67 \times 10^{-3}$ | $-6{\cdot}27 \times 10^{-1}$ | $5{\cdot}87 \times 10^{-4}$ |
| 1000 | 3 | $-2{\cdot}22$ | $1{\cdot}93 \times 10^{-3}$ | $-2{\cdot}14$ | $3{\cdot}38 \times 10^{-4}$ |
| 1000 | 9 | $-7{\cdot}56$ | $2{\cdot}00 \times 10^{-5}$ | $-6{\cdot}21$ | $5{\cdot}10 \times 10^{-4}$ |
| 1000 | 10 | $-8{\cdot}46$ | $9{\cdot}32 \times 10^{-6}$ | $-6{\cdot}60$ | $6{\cdot}19 \times 10^{-4}$ |
| 1000 | 15 | $-1{\cdot}31 \times 10^{1}$ | $4{\cdot}71 \times 10^{-7}$ | $1{\cdot}89$ | $3{\cdot}58 \times 10^{-3}$ |
| 1000 | 16 | $-1{\cdot}40 \times 10^{1}$ | $7{\cdot}91 \times 10^{-7}$ | $3{\cdot}04 \times 10^{1}$ | $1{\cdot}03 \times 10^{-2}$ |
| 5000 | 1 | $-7{\cdot}65 \times 10^{-1}$ | $1{\cdot}53 \times 10^{-3}$ | $-7{\cdot}80 \times 10^{-1}$ | $8{\cdot}41 \times 10^{-5}$ |
| 10000 | 1 | $-8{\cdot}12 \times 10^{-1}$ | $8{\cdot}99 \times 10^{-4}$ | $-8{\cdot}25 \times 10^{-1}$ | $3{\cdot}82 \times 10^{-5}$ |
| 50000 | 1 | $-8{\cdot}89 \times 10^{-1}$ | $2{\cdot}72 \times 10^{-4}$ | $-8{\cdot}97 \times 10^{-1}$ | $6{\cdot}41 \times 10^{-6}$ |
| 100000 | 1 | $-9{\cdot}12 \times 10^{-1}$ | $1{\cdot}55 \times 10^{-4}$ | $-9{\cdot}19 \times 10^{-1}$ | $2{\cdot}68 \times 10^{-6}$ |

TABLE 2. Values of $R$, $\alpha$, $-\alpha c_r$, $A$, $\omega$ and $E$.

i.e. we integrate the sum of the squares of the perturbations with respect to $y$ and one period of $\theta$ and then divide by the length of the intervals. To first order in $A^2$

$$E = A^2 \int_{-1}^{1} \left[\frac{1}{\alpha^2} \,|D\phi^{(1;\,1)}|^2 + |\phi^{(1;\,1)}|^2\right] dy.$$

As in experiments, the larger the Reynolds number, the smaller the size of the disturbance. For a given fixed $R$, the minimum $E$ occurs for fairly small $\alpha$. For small $\alpha$, the frequency $\omega$ is very close to the linear frequency $-\alpha c_r$. As $\alpha$ increases, the frequency becomes more and more distorted from the linear case, and a larger disturbance is necessary to create a steady-state solution. Finally a region is reached where there are no steady-state solutions.

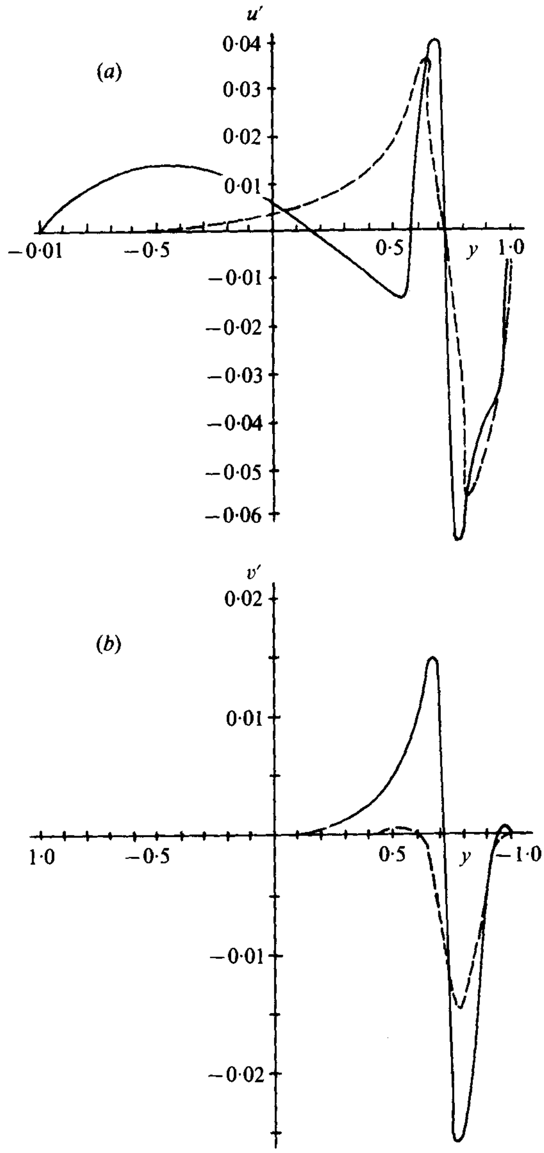


FIGURE 2. The linear solution (dashed curve) and the nonlinear solution (solid curve) for (*a*) $u'$ and (*b*) $v'$ when $R = 1000$, $\alpha = 3$ and $\theta = 0$.

Ellingsen *et al.* solved this problem for the cases $\alpha = 0{\cdot}5$ and $\alpha = 1$ for $R$ varying from 1000 to 10000. They solved the linear eigenvalue problem using asymptotic expansions. Their eigenvalues agree with ours to three decimal places. Also, they assumed that $\omega = -\alpha c_r$. This is approximately correct if $\alpha$ is small.

They found the nonlinear terms to be destabilizing, which checks our results. Normalizing the vorticity, they found that $A$ decreases very slowly with increasing $R$, and seemingly approaches a constant value.

With our normalization, $A$ and $E$ decrease fairly rapidly. In fact, if $\alpha = 1$, $E$ is approximately proportional to $R^{-\frac{1}{3}}$. For $R \leqslant 1000$, $E \approx 5{\cdot}8R^{-\frac{1}{3}}$. The constant then

| $R$ | $A$ | $\omega$ |
|---|---|---|
| 100 | $3{\cdot}92 \times 10^{-2}$ | $-1{\cdot}63 \times 10^{-1}$ |
| 200 | $2{\cdot}14 \times 10^{-2}$ | $-2{\cdot}60 \times 10^{-1}$ |
| 500 | $9{\cdot}70 \times 10^{-3}$ | $-4{\cdot}37 \times 10^{-1}$ |
| 700 | $7{\cdot}54 \times 10^{-3}$ | $-5{\cdot}04 \times 10^{-1}$ |

TABLE 3. Values of $R$, $A$ and $\omega$ for $\alpha = 1$ and a fourth-order approximation.

increases slowly, so that for $R = 100\,000$, $E = 12{\cdot}44R^{-\frac{1}{3}}$. So $E$ decreases steadily with increasing $R$, indicating decreasing stability with respect to finite disturbances.

The actual solutions can be easily obtained. Using a second-order approximation,

$$u = y + (2/\alpha)\,A[D(\operatorname{Re}\phi^{(1;\,1)})\cos\theta - D(\operatorname{Im}\phi^{(1;\,1)})\sin\theta] + (2/\alpha)\,A^2[D\phi^{(0;\,2)} + D(\operatorname{Re}\phi^{(2;\,2)})\cos 2\theta - D(\operatorname{Im}\phi^{(2;\,2)})\sin 2\theta], \tag{7.1}$$

$$v = 2A[(\operatorname{Re}\phi^{(1;\,1)})\sin\theta + (\operatorname{Im}\phi^{(1;\,1)})\cos\theta] + 4A^2[(\operatorname{Re}\phi^{(2;\,2)})\sin 2\theta + (\operatorname{Im}\phi^{(2;\,2)})\cos 2\theta]. \tag{7.2}$$

If we ignore the $A^2$ terms, this is the Orr–Sommerfeld solution.

The resulting linear and nonlinear perturbations $u'$ and $v'$ are plotted for the case $R = 1000$, $\alpha = 3$ and $\theta = 0$ in figure 2. For the linear case, the disturbance is concentrated near the boundary $y = 1$. As $R$ decreases or $\alpha$ increases, the disturbance becomes larger in magnitude, but retains the same basic shape. The nonlinear disturbance is slightly larger and more spread out. As $\alpha$ increases, the nonlinear solution becomes much larger than the linear perturbation.

The fourth-order solution is much harder to compute accurately. However, the only major change is in computing $\phi^{(1;\,3)}$. This function is not unique, since any multiple of $\phi^{(1;\,1)}$ can be added. We have used the normalization $\phi^{(1;\,3)}(0) = 1$. Table 3 gives a few solutions for $\alpha = 1$ and $R$ small. In these cases, the fourth-order terms cause the amplitude and the frequency to decrease in magnitude. So these terms are further destabilizing. Also, as $R$ increases, the solutions get closer to the second-order solutions.

Our results for small $R$, then, are not terribly accurate. However, they do seem to indicate the general behaviour of a nonlinear solution.

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